

# Covariant central extensions of gauge Lie algebras

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## Abstract

Motivated by positive energy representations, we classify those continuous central extensions of the compactly supported gauge Lie algebra that are covariant under a 1-parameter group of transformations of the base manifold.

## 1 Introduction

Let  $\pi: \mathcal{K} \rightarrow M$  be a locally trivial bundle of finite dimensional Lie groups, with corresponding Lie algebra bundle  $\mathfrak{K} \rightarrow M$ . We assume that the fibres  $\mathfrak{K}_x$  are semisimple. The group  $G = \Gamma_c(\mathcal{K})$  of compactly supported sections, called the (compactly supported) *gauge group*, is a locally convex Lie group with Lie algebra  $\mathfrak{g} = \Gamma_c(\mathfrak{K})$ , the (compactly supported) *gauge Lie algebra*.

In representation theory, one often wishes to impose *positive energy conditions* derived from a distinguished 1-parameter group  $\gamma_M: \mathbb{R} \rightarrow \text{Diff}(M)$  of transformations of the base. A lift  $\gamma: \mathbb{R} \rightarrow \text{Aut}(\mathcal{K})$  of  $\gamma_M$  induces a 1-parameter family  $\alpha: \mathbb{R} \rightarrow \text{Aut}(G)$  of automorphisms of the gauge group. If  $D \in \text{der}(\mathfrak{g})$  is the derivation  $D(\xi) := \frac{d}{dt}\big|_{t=0} \alpha_{t*}(\xi)$  induced by  $\alpha$ , then the semidirect product

$$G \rtimes_{\alpha} \mathbb{R}$$

is a locally convex Lie group with Lie algebra

$$\mathfrak{g} \rtimes_D \mathbb{R}.$$

Since  $[0 \oplus 1, \xi \oplus 0] = D(\xi)$ , we will identify  $0 \oplus 1$  with  $D$  and write  $\mathfrak{g} \rtimes_D \mathbb{R} = \mathfrak{g} \rtimes \mathbb{R}D$  accordingly. In this note, we give a complete classification of the continuous 1-dimensional central extensions  $\widehat{\mathfrak{g}}$  of  $\mathfrak{g} \rtimes_D \mathbb{R}$ , in other words, we determine the continuous second Lie algebra cohomology  $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ .

In order to describe the answer, write  $\mathbf{v} \in \mathcal{V}(\mathcal{K})$  for the vector field on  $\mathcal{K}$  that generates the flow of  $\gamma$ , and write  $\pi_* \mathbf{v} \in \mathcal{V}(M)$  for its projection to  $M$ ,

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which generates the flow of  $\gamma_M$ . Identifying  $\xi \in \Gamma_c(\mathfrak{K})$  with the corresponding vertical left invariant vector field  $\Xi_\xi$  on  $\mathcal{K}$ , the action of the derivation  $D$  on  $\mathfrak{g} = \Gamma_c(\mathfrak{K})$  is described by  $D\xi = L_{\mathbf{v}}\xi$ . For each fibre  $\mathfrak{K}_x$ , the universal invariant bilinear form  $\kappa$  takes values in the  $K$ -representation  $V(\mathfrak{K}_x)$ , and  $\mathbb{V} := V(\mathfrak{K})$  is a flat bundle over  $M$ . In the (important!) special case that  $\mathfrak{K}_x$  is a compact simple Lie algebra,  $\kappa$  is simply the Killing form with values in  $V(\mathfrak{K}_x) = \mathbb{R}$ , and  $\mathbb{V}$  is the trivial real line bundle over  $M$ . Given a Lie connection  $\nabla$  on  $\mathfrak{K}$  and a closed  $\pi_*\mathbf{v}$ -invariant current  $\lambda \in \Omega_c^1(M, \mathbb{V})'$ , there is a unique 2-cocycle  $\omega_{\lambda, \nabla}$  on  $\mathfrak{g} \rtimes \mathbb{R}D$  with

$$\omega_{\lambda, \nabla}(\xi, \eta) = \lambda(\kappa(\xi, \nabla\eta)), \quad \omega_{\lambda, \nabla}(D, \xi) = \lambda(\kappa(L_{\mathbf{v}}\nabla, \xi)) \quad \text{for } \xi, \eta \in \mathfrak{g}.$$

The class  $[\omega_{\lambda, \nabla}] \in H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$  is independent of the choice of  $\nabla$ . One of our main results (Theorem 5.3) asserts that the map  $\lambda \mapsto [\omega_{\lambda, \nabla}]$  is a linear isomorphism from the space of closed,  $\pi_*\mathbf{v}$ -invariant,  $\mathbb{V}$ -valued currents on  $M$  to the continuous Lie algebra cohomology  $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ .

Our motivation for classifying these central extensions comes from the theory of *projective positive energy representations*. If  $G$  is a Lie group with locally convex Lie algebra  $\mathfrak{g}$ , and  $\alpha: \mathbb{R} \rightarrow \text{Aut}(G)$  is a homomorphism defining a smooth  $\mathbb{R}$ -action on  $G$ , then the semidirect product  $G \rtimes_\alpha \mathbb{R}$  is again a Lie group, with Lie algebra  $\mathfrak{g} \rtimes \mathbb{R}D$ . For every smooth projective unitary representation  $\bar{\rho}: G \rtimes_\alpha \mathbb{R} \rightarrow \text{PU}(\mathcal{H})$  of  $G \rtimes_\alpha \mathbb{R}$ , there exists a central Lie group extension  $\widehat{G}$  of  $G \rtimes_\alpha \mathbb{R}$  by the circle group  $\mathbb{T}$  for which  $\bar{\rho}$  lifts to a smooth *linear* unitary representation  $\rho: \widehat{G} \rightarrow \text{U}(\mathcal{H})$  (see [JN15] for details). The Lie algebra  $\widehat{\mathfrak{g}}$  can then be written as

$$\widehat{\mathfrak{g}} = \mathbb{R}C \oplus_\omega (\mathfrak{g} \rtimes \mathbb{R}D), \tag{1}$$

where  $\omega$  is a Lie algebra 2-cocycle of  $\mathfrak{g} \rtimes \mathbb{R}D$ . The Lie bracket is

$$[zC + x + tD, z'C + x' + t'D] = \omega(x + tD, x' + t'D)C + [x, x'] + tD(x') - t'D(x),$$

and  $d\rho(C) = i\mathbf{1}$  by construction. We say that  $\bar{\rho}$  is a *positive energy representation* if the selfadjoint operator  $H := i d\rho(D)$  has a spectrum which is bounded below.

In [JN16] we address the problem of classifying the projective positive energy representations of the gauge group  $G = \Gamma_c(\mathcal{K})$ , for the smooth action  $\alpha: \mathbb{R} \rightarrow \text{Aut}(G)$  induced by a smooth 1-parameter group  $\gamma: \mathbb{R} \rightarrow \text{Aut}(\mathcal{K})$  of bundle automorphisms. We break this problem into the following steps:

- (PE1) Classify the 1-dimensional central Lie algebra extensions  $\widehat{\mathfrak{g}}$  of  $\mathfrak{g} \rtimes_D \mathbb{R}$ .
- (PE2) Determine which central extensions  $\widehat{\mathfrak{g}}$  fulfill natural positivity conditions imposed by so-called Cauchy–Schwarz estimates required for cocycles coming from positive energy representations (cf. [JN16]).
- (PE3) For those  $\widehat{\mathfrak{g}}$ , classify the positive energy representations that integrate to a representation of a connected Lie group  $\widehat{G}_0$  with Lie algebra  $\widehat{\mathfrak{g}}$ .

In the present note we completely solve (PE1) for semisimple structure algebras  $\mathfrak{K}_x$ , thus completing the first step in the classification of projective positive energy representations.

To proceed with (PE2), we assume in [JN16] that the vector field  $\pi_*\mathbf{v}$  on  $M$  has no zeros and generates a periodic flow, hence defines an action of the circle group  $\mathbb{T}$  on  $M$ . Under this assumption we then show that for every projective positive energy representation  $\overline{\rho}$  of  $\mathfrak{g} \rtimes \mathbb{R}D$ , there exists a locally finite set  $\Lambda \subseteq M/\mathbb{T}$  of orbits such that the  $\mathfrak{g}$ -part of  $d\overline{\rho}$  factors through the restriction homomorphism

$$\mathfrak{g} = \Gamma_c(\mathfrak{K}) \rightarrow \Gamma_c(\mathfrak{K}|_{\Lambda_M}) \cong \bigoplus_{\lambda \in \Lambda} \mathcal{L}_{\psi_\lambda}(\mathfrak{k}), \quad (2)$$

where  $\Lambda_M \subseteq M$  is the union of the orbits in  $M$ , and

$$\mathcal{L}_\psi(\mathfrak{k}) = \{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall t \in \mathbb{R}) \xi(t+1) = \psi^{-1}(\xi(t))\}$$

is the loop algebra twisted by a finite order automorphism  $\psi \in \text{Aut}(\mathfrak{k})$ . As the positive energy representations of covariant loop algebras and their central extensions, the Kac–Moody algebras ([Ka85]), are well understood ([PS86]), this allows us to solve (PE3). This result contributes in particular to “non-commutative distribution” program whose goal is a classification of the irreducible unitary representations of gauge groups ([A-T93]).

The structure of this paper is as follows. After introducing gauge groups, their Lie algebras and one-parameter groups of automorphism in Section 2, we describe in Section 3 a procedure that provides a reduction from semisimple to simple structure Lie algebras, at the expense of replacing  $M$  by a finite covering manifold  $\tilde{M}$ . In Section 4, we introduce the flat bundle  $\mathbb{V}$ , which is used in a crucial way in Section 5 for the description of the natural 2-cocycles on the gauge algebra. The first step (PE1) is completely settled in Section 5, where Theorem 5.3 describes all 1-dimensional central extensions of the gauge algebra.

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## 2 Gauge groups and gauge algebras

Let  $\mathcal{K} \rightarrow M$  be a smooth bundle of Lie groups, and let  $\mathfrak{K} \rightarrow M$  be the associated Lie algebra bundle with fibres  $\mathfrak{K}_x = \text{Lie}(\mathcal{K}_x)$ . If  $M$  is connected, then the fibres  $\mathcal{K}_x$  of  $\mathcal{K} \rightarrow M$  are all isomorphic to a fixed structure group  $K$ , and the fibres  $\mathfrak{K}_x$  of  $\mathfrak{K}$  are isomorphic to its Lie algebra  $\mathfrak{k} = \text{Lie}(K)$ .

**Definition 2.1.** (Gauge group) The *gauge group* is the group  $\Gamma(\mathcal{K})$  of smooth sections of  $\mathcal{K} \rightarrow M$ , and the *compactly supported gauge group* is the group  $\Gamma_c(\mathcal{K})$  of smooth compactly supported sections.

**Definition 2.2.** (Gauge algebra) The *gauge algebra* is the Fréchet-Lie algebra  $\Gamma(\mathfrak{K})$  of smooth sections of  $\mathfrak{K} \rightarrow M$ , equipped with the pointwise Lie bracket. The *compactly supported gauge algebra*  $\Gamma_c(\mathfrak{K})$  is the LF-Lie algebra of smooth compactly supported sections.

The compactly supported gauge group  $\Gamma_c(\mathcal{K})$  is a locally convex Lie group, whose Lie algebra is the compactly supported gauge algebra  $\Gamma_c(\mathfrak{K})$ .

**Proposition 2.3.** *There exists a unique smooth structure on  $\Gamma_c(\mathcal{K})$  which makes it a locally exponential Lie group with Lie algebra  $\Gamma_c(\mathfrak{K})$  and exponential map  $\exp: \Gamma_c(\mathfrak{K}) \rightarrow \Gamma_c(\mathcal{K})$  defined by pointwise exponentiation.*

*Proof.* It suffices to prove this in the case that  $M$  is connected. Let  $V_{\mathfrak{k}}, W_{\mathfrak{k}} \subseteq \mathfrak{k}$  be open, symmetric 0-neighbourhoods such that the exponential  $\exp: \mathfrak{k} \rightarrow K$  restricts to a diffeomorphism of  $W_{\mathfrak{k}}$  onto its image,  $V_{\mathfrak{k}}$  is contained in  $W_{\mathfrak{k}}$ , and  $\exp(V_{\mathfrak{k}}) \cdot \exp(V_{\mathfrak{k}}) \subseteq \exp(W_{\mathfrak{k}})$ .

Choose a locally finite cover  $(U_i)_{i \in I}$  of  $M$  by open trivialising neighbourhoods for  $\mathcal{K} \rightarrow M$ , which possesses a refinement  $(C_i)_{i \in I}$  such that  $C_i \subset U_i$  is compact for all  $i \in I$ . Fix local trivialisations  $\varphi_i: K \times U_i \xrightarrow{\sim} \mathcal{K}|_{U_i}$  of  $\mathcal{K}$ , which gives rise to local trivialisations  $d\varphi_i: \mathfrak{k} \times U_i \xrightarrow{\sim} \mathfrak{K}|_{U_i}$  for  $\mathfrak{K}$ . Define  $W_i := d\varphi_i(U_i \times W_K)$ , and set

$$W_{\Gamma_c(\mathfrak{K})} := \{\xi \in \Gamma_c(\mathfrak{K}); \xi(C_i) \subseteq W_i \forall i \in I\}.$$

Similarly,  $V_{\Gamma_c(\mathfrak{K})}$  is defined in terms of preimages over  $C_i$  of  $V_i := d\varphi_i(U_i \times V_K)$ , and both  $V_{\Gamma_c(\mathfrak{K})}$  and  $W_{\Gamma_c(\mathfrak{K})}$  are open in  $\Gamma_c(\mathfrak{K})$ . Since the pointwise exponential  $\exp: \Gamma_c(\mathfrak{K}) \rightarrow \Gamma_c(\mathcal{K})$  is a bijection of  $W_{\Gamma_c(\mathfrak{K})}$  onto its image  $W_{\Gamma_c(\mathcal{K})} := \exp(W_{\Gamma_c(\mathfrak{K})})$ , the latter inherits a smooth structure. The same goes for its subset  $V_{\Gamma_c(\mathcal{K})} := \exp(V_{\Gamma_c(\mathfrak{K})})$ .

Inversion  $W_{\Gamma_c(\mathcal{K})} \rightarrow W_{\Gamma_c(\mathcal{K})}$  and multiplication  $V_{\Gamma_c(\mathcal{K})} \times V_{\Gamma_c(\mathcal{K})} \rightarrow W_{\Gamma_c(\mathcal{K})}$  are smooth, and for every  $\sigma \in \Gamma_c(\mathcal{K})$ , there exists an open 0-neighbourhood  $W_{\sigma} \subseteq W_{\Gamma_c(\mathfrak{K})}$  such that  $\text{Ad}_{\sigma}: W_{\sigma} \rightarrow W_{\Gamma_c(\mathfrak{K})}$  is smooth. It therefore follows from [Ti83, p.14] (which generalises to locally convex Lie groups, cf. [Ne06, Thm. II.2.1]), that  $\Gamma_c(\mathcal{K})$  possesses a unique Lie group structure such that for some open 0-neighbourhood  $U_{\Gamma_c(\mathfrak{K})} \subseteq W_{\Gamma_c(\mathfrak{K})}$ , the image  $\exp(U_{\Gamma_c(\mathfrak{K})}) \subseteq \Gamma_c(\mathcal{K})$  is an open neighbourhood of the identity.  $\square$

**Example 2.4.** If  $\mathcal{K} \rightarrow M$  is a trivial bundle, then the gauge group is  $\Gamma(\mathcal{K}) = C^\infty(M, K)$ , and the gauge algebra is  $\Gamma(\mathfrak{K}) = C^\infty(M, \mathfrak{k})$ . Similarly, we have  $\Gamma_c(\mathcal{K}) = C_c^\infty(M, K)$  and  $\Gamma_c(\mathfrak{K}) = C_c^\infty(M, \mathfrak{k})$  for their compactly supported versions. One can thus think of gauge groups as ‘twisted versions’ of the group of smooth  $K$ -valued functions on  $M$ .

The motivating example of a gauge group is the group  $\text{Gau}(P)$  of vertical automorphisms of a principal fibre bundle  $\pi: P \rightarrow M$  with structure group  $K$ .

**Example 2.5.** (Gauge groups from principal bundles) A *vertical automorphism* of a principal fibre bundle  $\pi: P \rightarrow M$  is a  $K$ -equivariant diffeomorphism  $\alpha: P \rightarrow P$  such that  $\pi \circ \alpha = \pi$ . The group  $\text{Gau}(P)$  of vertical automorphisms is called the *gauge group* of  $P$ . It is isomorphic to the group

$$C^\infty(P, K)^K := \{f \in C^\infty(P, K); (\forall p \in P, k \in K) f(pk) = k^{-1}f(p)k\}, \quad (3)$$

with isomorphism  $C^\infty(P, K)^K \xrightarrow{\sim} \text{Gau}(P)$  given by  $f \mapsto \alpha_f$  with  $\alpha_f(p) = pf(p)$ . In order to interpret  $\text{Gau}(P)$  as a gauge group in the sense of Definition 2.1, we construct the bundle of groups  $\text{Conj}(P) \rightarrow M$  with typical fibre  $K$ . For an element  $k \in K$ , we write  $c_k(g) = kgk^{-1}$  for the induced inner automorphism of  $K$ , and also  $\text{Ad}_k \in \text{Aut}(\mathfrak{k})$  for the corresponding automorphism of its Lie algebra  $\mathfrak{k}$ . Define the bundle of groups  $\text{Conj}(P) \rightarrow M$  by

$$\text{Conj}(P) := P \times K / \sim,$$

where  $\sim$  is the relation  $(pk, h) \sim (p, c_k(h))$  for  $p \in P$  and  $k, h \in K$ . We then have isomorphisms

$$\text{Gau}(P) \simeq C^\infty(P, K)^K \simeq \Gamma(\text{Conj}(P)),$$

where  $f \in C^\infty(P, K)^K$  corresponds to the section  $\sigma_f \in \Gamma(\text{Conj}(P))$  defined by  $\sigma_f(\pi(p)) = [p, f(p)]$  for all  $p \in P$ . The bundle of Lie algebras associated to  $\text{Conj}(P)$  is the *adjoint bundle*  $\text{Ad}(P) \rightarrow M$ , defined as the quotient

$$\text{Ad}(P) := P \times_{\text{Ad}} \mathfrak{k}$$

of  $P \times \mathfrak{k}$  modulo the relation  $(pk, X) \sim (p, \text{Ad}_k(X))$  for  $p \in P$ ,  $X \in \mathfrak{k}$  and  $k \in K$ . The *compactly supported gauge group*  $\text{Gau}_c(P) \subseteq \text{Gau}(P)$  is the group of vertical bundle automorphisms of  $P$  that are trivial outside the preimage of some compact set in  $M$ . Since it is isomorphic to  $\Gamma_c(\text{Conj}(P))$ , it is a locally convex Lie group with Lie algebra  $\mathfrak{gau}_c(P) = \Gamma_c(\text{Ad}(P))$ .

**Remark 2.6.** Gauge groups arise in field theory, as groups of transformations of the space of principal connections on  $P$  (the gauge fields). If the space-time manifold  $M$  is not compact, then one imposes boundary conditions on the gauge fields at infinity. Depending on how one does this, the group  $\text{Gau}(P)$  may be too big to preserve the set of admissible gauge fields. One then expects the group of remaining gauge transformations to at least contain  $\text{Gau}_c(P)$ , or perhaps even some larger Lie group of gauge transformations specified by a decay condition at infinity (cf. [Wa10, Go04]).

An *automorphism* of  $\pi: \mathcal{K} \rightarrow M$  is a pair  $(\gamma, \gamma_M) \in \text{Diff}(\mathcal{K}) \times \text{Diff}(M)$  with  $\pi \circ \gamma = \gamma_M \circ \pi$ , such that for each fibre  $\mathcal{K}_x$ , the map  $\gamma|_{\mathcal{K}_x}: \mathcal{K}_x \rightarrow \mathcal{K}_{\gamma_M(x)}$  is a group homomorphism. Since  $\gamma_M$  is determined by  $\gamma$ , we will omit it from the notation. We denote the group of automorphisms of  $\mathcal{K}$  by  $\text{Aut}(\mathcal{K})$ .

**Definition 2.7.** (Geometric  $\mathbb{R}$ -actions) In the context of gauge groups, we will be interested in  $\mathbb{R}$ -actions  $\alpha: \mathbb{R} \rightarrow \text{Aut}(\Gamma(\mathcal{K}))$  that are of *geometric* type. These are derived from a 1-parameter group  $\gamma: \mathbb{R} \rightarrow \text{Aut}(\mathcal{K})$  by

$$\alpha_t(\sigma) := \gamma_t \circ \sigma \circ \gamma_{M,t}^{-1}. \quad (4)$$

**Remark 2.8.** If  $\mathcal{K}$  is of the form  $\text{Ad}(P)$  for a principal fibre bundle  $P \rightarrow M$ , then a 1-parameter group of automorphisms of  $P$  induces a 1-parameter group of automorphisms of  $\mathcal{K}$ . If we think of the induced diffeomorphisms  $\gamma_M(t) \in \text{Diff}(M)$  as time translations, then the automorphisms of  $P$  encode the time translation behaviour of the gauge fields.

The 1-parameter group  $\alpha: \mathbb{R} \rightarrow \text{Aut}(\Gamma(\mathcal{K}))$  of group automorphisms differentiates to a 1-parameter group  $\beta: \mathbb{R} \rightarrow \text{Aut}(\Gamma(\mathfrak{K}))$  of Lie algebra automorphisms given by

$$\beta_t(\xi) = \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \gamma_t \circ e^{\varepsilon \xi} \circ \gamma_{M,t}^{-1}. \quad (5)$$

The corresponding derivation  $D := \frac{\partial}{\partial t} \Big|_{t=0} \beta_t$  of  $\Gamma(\mathfrak{K})$  can be described in terms of the infinitesimal generator  $\mathbf{v} \in \mathfrak{X}(\mathcal{K})$  of  $\gamma$ , given by  $\mathbf{v} := \frac{\partial}{\partial t} \Big|_{t=0} \gamma_t$ . We identify  $\xi \in \Gamma(\mathfrak{K})$  with the vertical, left invariant vector field  $\Xi_\xi \in \mathfrak{X}(\mathcal{K})$  defined by  $\Xi_\xi(k_x) = \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} k_x e^{-\varepsilon \xi(x)}$ . Using the equality  $[\mathbf{v}, \Xi_\xi] = \Xi_{D(\xi)}$ , we write

$$D(\xi) = L_{\mathbf{v}} \xi. \quad (6)$$

For  $\mathfrak{g} = \Gamma_c(\mathfrak{K})$ , the Lie algebra  $\mathfrak{g} \rtimes_D \mathbb{R}$  then has bracket

$$[\xi \oplus t, \xi' \oplus t'] = ([\xi, \xi'] + (tL_{\mathbf{v}}\xi' - t'L_{\mathbf{v}}\xi)) \oplus 0. \quad (7)$$

### 3 Reduction to simple Lie algebras

In this note, we will focus attention on the class of gauge algebras with a semisimple structure group, not only because they are more accessible, but also because they are relevant in applications. We now show that every gauge algebra with a *semisimple* structure group can be considered as a gauge algebra of a bundle with a *simple* structure group which need not be the same for all fibers. Accordingly, the base manifold  $M$  is replaced by a not necessarily connected finite cover.

#### 3.1 From semisimple to simple Lie algebras

Let  $\mathfrak{K} \rightarrow M$  be a smooth locally trivial bundle of Lie algebras with semisimple fibres. We construct a finite cover  $\widehat{M} \rightarrow M$  and a locally trivial bundle of Lie algebras  $\widehat{\mathfrak{K}} \rightarrow \widehat{M}$  with simple fibres such that  $\Gamma(\mathfrak{K}) \simeq \Gamma(\widehat{\mathfrak{K}})$  and  $\Gamma_c(\mathfrak{K}) \simeq \Gamma_c(\widehat{\mathfrak{K}})$ .

Because one can go back and forth between principal fibre bundles and bundles of Lie algebras, this shows that every gauge algebra for a principal fibre bundle with semisimple structure group is isomorphic to one with a simple structure group. Indeed, every principal fibre bundle  $P \rightarrow M$  with semisimple structure group  $K_i$  over the connected component  $M_i$  of  $M$  gives rise to the bundle  $\text{Ad}(P) \rightarrow M$  of Lie algebras. Conversely, every Lie algebra bundle  $\mathfrak{K} \rightarrow M$  with semisimple structure algebra  $\mathfrak{k}_i$  over the  $M_i$  gives rise to a principal fibre bundle  $P_{\mathfrak{K}} \rightarrow M$  with semisimple structure group  $\text{Aut}(\mathfrak{k}_i)$  over  $M_i$  defined, for  $x \in M_i$ , by  $P_{\mathfrak{K},x} := \text{Iso}(\mathfrak{k}_i, \mathfrak{K}_x)$  for  $x \in M_i$ .

**Theorem 3.1.** (Reduction from semisimple to simple structure algebras) *If  $\mathfrak{K} \rightarrow M$  is a smooth locally trivial bundle of Lie algebras with semisimple fibres, then there exists a finite cover  $\widehat{M} \rightarrow M$  and a smooth locally trivial bundle of Lie algebras  $\widehat{\mathfrak{K}} \rightarrow \widehat{M}$  with simple fibres such that there exist isomorphisms  $\Gamma(\mathfrak{K}) \simeq \Gamma(\widehat{\mathfrak{K}})$  and  $\Gamma_c(\mathfrak{K}) \simeq \Gamma_c(\widehat{\mathfrak{K}})$  of locally convex Lie algebras.*

The finite cover  $\widehat{M} \rightarrow M$  is not necessarily connected, and the isomorphism classes of the fibres of  $\widehat{\mathfrak{K}} \rightarrow \widehat{M}$  are not necessarily the same over different connected components of  $\widehat{M}$ .

*Proof.* For a finite dimensional semisimple Lie algebra  $\mathfrak{k}$ , we write  $\text{Spec}(\mathfrak{k})$  for the finite set of maximal ideals of  $\mathfrak{k}$ , equipped with the discrete topology. We now define the set

$$\widehat{M} := \bigcup_{x \in M} \text{Spec}(\mathfrak{K}_x)$$

with the natural projection  $\text{pr}_{\widehat{M}}: \widehat{M} \rightarrow M$ . Local trivialisations  $\mathfrak{K}|_U \simeq U \times \mathfrak{k}$  of  $\mathfrak{K}$  over open connected subsets  $U \subseteq M$  induce compatible bijections between  $\text{pr}_{\widehat{M}}^{-1}(U)$  and the smooth manifold  $U \times \text{Spec}(\mathfrak{k})$ . This provides  $\widehat{M}$  with a manifold structure for which  $\text{pr}_{\widehat{M}}: \widehat{M} \rightarrow M$  is a finite covering.<sup>1</sup> We define

$$\widehat{\mathfrak{K}} := \bigcup_{I_x \in \widehat{M}} \mathfrak{K}_x / I_x$$

with the natural projection  $\pi: \widehat{\mathfrak{K}} \rightarrow \widehat{M}$ . Local trivialisations  $\mathfrak{K}|_U \simeq U \times \mathfrak{k}$  of  $\mathfrak{K}$  yield bijections between  $\widehat{\mathfrak{K}}|_U$  and the disjoint union

$$\bigsqcup_{I \in \text{Spec}(\mathfrak{k})} U_I \times (\mathfrak{k}/I),$$

where  $U_I \simeq U$  is the connected component of  $\text{pr}_{\widehat{M}}^{-1}(U)$  corresponding to the maximal ideal  $I \subseteq \mathfrak{k}$  in the particular trivialisation. Since different trivialisations differ by Lie algebra automorphisms of the fibres, which permute the ideals in

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<sup>1</sup>Note that non-isomorphic maximal ideals of  $\mathfrak{K}_x$  are always in different connected components of  $\widehat{M}$ , whereas isomorphic maximal ideals may or may not be in the same connected component, depending on the bundle structure.

$U_I$  and  $\mathfrak{k}/I$  alike, the projection  $\pi: \widehat{\mathfrak{K}} \rightarrow \widehat{M}$  becomes a smooth locally trivial bundle of Lie algebras over  $\widehat{M}$ .

The morphism  $\Phi: \Gamma(\mathfrak{K}) \rightarrow \Gamma(\widehat{\mathfrak{K}})$  of Fréchet Lie algebras defined by

$$\Phi(\sigma)(I_x) := \sigma(x) + I_x$$

is an isomorphism; because the fibres are semisimple, the injection  $\mathfrak{K}_x/I_x \hookrightarrow \mathfrak{K}_x$  allows one to construct the inverse

$$\Phi^{-1}(\tau)(x) = \sum_{I_x \in \text{Spec}(\mathfrak{K}_x)} \tau(I_x).$$

Since the projection  $\text{pr}_{\widehat{M}}: \widehat{M} \rightarrow M$  is a finite cover, this induces an isomorphism  $\Phi: \Gamma_c(\mathfrak{K}) \rightarrow \Gamma_c(\widehat{\mathfrak{K}})$  of LF-Lie algebras.  $\square$

Clearly, a smooth 1-parameter family of automorphisms of  $\mathfrak{K} \rightarrow M$  acts naturally on the maximal ideals, so we obtain a smooth action on  $\widehat{M} \rightarrow M$  and on  $\widehat{\mathfrak{K}} \rightarrow \widehat{M}$ . The action on  $\widehat{M}$  is locally free or periodic if and only if the action on  $M$  is, and then the period on  $\widehat{M}$  is a multiple of the period on  $M$ .

**Example 3.2.** If  $\mathfrak{k}$  is a simple Lie algebra, then  $\widehat{M} = M$ .

**Example 3.3.** If  $P = M \times K$  is trivial, then  $\widehat{M} = M \times \text{Spec}(\mathfrak{k})$  and all connected components of  $\widehat{M}$  are diffeomorphic to  $M$ .

**Example 3.4.** If  $\mathfrak{k}$  is a semisimple Lie algebra with  $r$  simple ideals that are mutually non-isomorphic, then  $\widehat{M} = \bigsqcup_{i=1}^r M$  is a disjoint union of copies of  $M$ .

**Example 3.5.** (Frame bundles of 4-manifolds) Let  $M$  be a 4-dimensional Riemannian manifold. Let  $P := \text{OF}(M)$  be the principal  $\text{O}(4, \mathbb{R})$ -bundle of orthogonal frames. Then  $\mathfrak{k} = \mathfrak{so}(4, \mathbb{R})$  is isomorphic to  $\mathfrak{su}_L(2, \mathbb{C}) \oplus \mathfrak{su}_R(2, \mathbb{C})$ . The group  $\pi_0(K)$  is of order 2, the non-trivial element acting by conjugation with  $T = \text{diag}(-1, 1, 1, 1)$ . Since this permutes the two simple ideals, the manifold  $\widehat{M}$  is the orientable double cover of  $M$ . This is the disjoint union  $\widehat{M} = \widehat{M}_L \sqcup \widehat{M}_R$  of two copies of  $M$  if  $M$  is orientable, and a connected twofold cover  $\widehat{M} \rightarrow M$  if it is not.

### 3.2 Compact and noncompact simple Lie algebras

A semisimple Lie algebra  $\mathfrak{k}$  is called *compact* if its Killing form is negative definite. Every semisimple Lie algebra  $\mathfrak{k}$  is a direct sum  $\mathfrak{k} = \mathfrak{k}_{\text{cpt}} \oplus \mathfrak{k}_{\text{nc}}$ , where  $\mathfrak{k}_{\text{cpt}}$  is the direct sum of all compact ideals of  $\mathfrak{k}$  (or, alternatively, its maximal compact quotient), and  $\mathfrak{k}_{\text{nc}}$  is the direct sum of the noncompact ideals. Since the decomposition  $\mathfrak{k} = \mathfrak{k}_{\text{cpt}} \oplus \mathfrak{k}_{\text{nc}}$  is invariant under  $\text{Aut}(\mathfrak{k})$ , every Lie algebra bundle  $\mathfrak{K} \rightarrow M$  can be written as a direct sum

$$\mathfrak{K} = \mathfrak{K}_{\text{cpt}} \oplus \mathfrak{K}_{\text{nc}} \tag{8}$$



of Lie algebra bundles over  $M$ . Correspondingly, we have the decomposition

$$\widehat{M} = \widehat{M}_{\text{cpt}} \sqcup \widehat{M}_{\text{nc}} \quad (9)$$

of  $\widehat{M}$  into disjoint submanifolds,  $\widehat{M}_{\text{cpt}}$  and  $\widehat{M}_{\text{nc}}$ , containing the maximal ideals  $I_x \subset \mathfrak{K}_x$  with  $\mathfrak{K}_x/I_x$  compact and noncompact, respectively. Writing  $\widehat{\mathfrak{K}}_{\text{cpt}}$  for the restriction of  $\widehat{\mathfrak{K}}$  to  $\widehat{M}_{\text{cpt}}$  and  $\widehat{\mathfrak{K}}_{\text{nc}}$  for its restriction to  $\widehat{M}_{\text{nc}}$ , we find Lie algebra bundles  $\widehat{\mathfrak{K}}_{\text{cpt}} \rightarrow \widehat{M}_{\text{cpt}}$  and  $\widehat{\mathfrak{K}}_{\text{nc}} \rightarrow \widehat{M}_{\text{nc}}$  with compact and noncompact simple fibres respectively, and Fréchet Lie algebra isomorphisms

$$\Gamma(\mathfrak{K}_{\text{cpt}}) \simeq \Gamma(\widehat{\mathfrak{K}}_{\text{cpt}}) \quad \text{and} \quad \Gamma(\mathfrak{K}_{\text{nc}}) \simeq \Gamma(\widehat{\mathfrak{K}}_{\text{nc}}). \quad (10)$$

## 4 Universal invariant symmetric bilinear forms

In Section 5, we will undertake a detailed analysis of the 2-cocycles of  $\mathfrak{g} \rtimes_D \mathbb{R}$  for compactly supported gauge algebras  $\mathfrak{g} := \Gamma_c(\mathfrak{K})$  with semisimple structure group  $K$ . In order to describe the relevant 2-cocycles, we need to introduce universal invariant symmetric bilinear forms on the Lie algebra  $\mathfrak{k}$  of the structure group. In the case that  $\mathfrak{k}$  is a compact simple Lie algebra, this is simply the Killing form.

### 4.1 Universal invariant symmetric bilinear forms

Let  $\mathfrak{k}$  be a finite dimensional Lie algebra. Then its automorphism group  $\text{Aut}(\mathfrak{k})$  is a closed subgroup of  $\text{GL}(\mathfrak{g})$ , hence a Lie group with Lie algebra  $\text{der}(\mathfrak{k})$ . Since  $\text{der}(\mathfrak{k})$  acts trivially on the quotient

$$V(\mathfrak{k}) := S^2(\mathfrak{k})/(\text{der}(\mathfrak{k}) \cdot S^2(\mathfrak{k}))$$

of the twofold symmetric tensor power  $S^2(\mathfrak{k})$ , the the  $\text{Aut}(\mathfrak{k})$ -representation on  $V(\mathfrak{k})$  factors through  $\pi_0(\text{Aut}(\mathfrak{k}))$ . The *universal  $\text{der}(\mathfrak{k})$ -invariant symmetric bilinear form* is defined by

$$\kappa: \mathfrak{k} \times \mathfrak{k} \rightarrow V(\mathfrak{k}), \quad \kappa(x, y) := [x \otimes_s y] = \frac{1}{2}[x \otimes y + y \otimes x].$$

We associate to  $\lambda \in V(\mathfrak{k})^*$  the  $\mathbb{R}$ -valued,  $\text{der}(\mathfrak{k})$ -invariant, symmetric, bilinear form  $\kappa_\lambda := \lambda \circ \kappa$ . This correspondence is a bijection between  $V(\mathfrak{k})^*$  and the space of  $\text{der}(\mathfrak{k})$ -invariant symmetric bilinear forms on  $\mathfrak{k}$ .

For the convenience of the reader, we now list some properties of  $V(\mathfrak{k})$  for (semi)simple Lie algebras  $\mathfrak{k}$ , in which case  $\text{der}(\mathfrak{k}) = \mathfrak{k}$ . These results will be used in the rest of the paper. We refer to [NW09, App. B] for proofs and a more detailed exposition.

For a simple real Lie algebra  $\mathfrak{k}$ , we have  $V(\mathfrak{k}) \simeq \mathbb{K}$ , with  $\mathbb{K} = \mathbb{C}$  if  $\mathfrak{k}$  admits a complex structure, and  $\mathbb{K} = \mathbb{R}$  if it does not, i.e., if  $\mathfrak{k}$  is *absolutely simple*. The universal invariant symmetric bilinear form can be identified with the Killing form of the real Lie algebra  $\mathfrak{k}$  if  $\mathbb{K} = \mathbb{R}$  and the Killing form of the underlying

complex Lie algebra if  $\mathbb{K} = \mathbb{C}$ . In particular, in the important special case that  $\mathfrak{k}$  is a compact simple Lie algebra, the universal invariant bilinear form  $\kappa: \mathfrak{k} \times \mathfrak{k} \rightarrow V(\mathfrak{k})$  is simply the negative definite Killing form  $\kappa: \mathfrak{k} \times \mathfrak{k} \rightarrow \mathbb{R}$ ,  $\kappa(x, y) = \text{tr}(\text{ad } x \text{ ad } y)$ .

For a semisimple real Lie algebra  $\mathfrak{k} = \bigoplus_{i=1}^r \mathfrak{k}_i^{m_i}$ , where the simple ideals  $\mathfrak{k}_i$  are mutually non-isomorphic, one finds  $V(\mathfrak{k}) \simeq \bigoplus_{i=1}^r V(\mathfrak{k}_i)^{m_i}$  with  $V(\mathfrak{k}_i)$  isomorphic to  $\mathbb{R}$  or  $\mathbb{C}$ . The action of  $\pi_0(\text{Aut}(\mathfrak{k}))$  on  $V(\mathfrak{k})$  leaves invariant the subspaces  $V(\mathfrak{k}_i)^{m_i}$  coming from the isotypical ideals  $\mathfrak{k}_i^{m_i}$ . If  $V(\mathfrak{k}_i) \simeq \mathbb{R}$ , then the action of  $\pi_0(\text{Aut}(\mathfrak{k}))$  on  $V(\mathfrak{k}_i)^{m_i}$  factors through the homomorphism  $\pi_0(\text{Aut}(\mathfrak{k})) \rightarrow S_{m_i}$  that maps  $\alpha \in \text{Aut}(\mathfrak{k})$  to the permutation it induces on the set of ideals isomorphic to  $\mathfrak{k}_i$ . If  $V(\mathfrak{k}_i) \simeq \mathbb{C}$ , then the action on  $\mathbb{C}^{m_i}$  factors through a homomorphism  $\pi_0(\text{Aut}(\mathfrak{k})) \rightarrow (\mathbb{Z}/2\mathbb{Z})^{m_i} \rtimes S_{m_i}$ , where the symmetric group  $S_{m_i}$  acts by permuting components and  $(\mathbb{Z}/2\mathbb{Z})^{m_i}$  acts by complex conjugation in the components.

## 4.2 The flat bundle $\mathbb{V} = V(\mathfrak{K})$

If  $\mathfrak{K} \rightarrow M$  is a bundle of Lie algebras, we denote by  $\mathbb{V} \rightarrow M$  the vector bundle with fibres  $\mathbb{V}_x = V(\mathfrak{K}_x)$ . It carries a canonical flat connection  $\mathbf{d}$ , defined by  $\mathbf{d}\kappa(\xi, \eta) := \kappa(\nabla\xi, \eta) + \kappa(\xi, \nabla\eta)$  for  $\xi, \eta \in \Gamma(\mathfrak{K})$ , where  $\nabla$  is a *Lie connection* on  $\mathfrak{K}$ , meaning that  $\nabla[\xi, \eta] = [\nabla\xi, \eta] + [\xi, \nabla\eta]$  for all  $\xi, \eta \in \Gamma(\mathfrak{K})$ . As any two Lie connections differ by a  $\text{der}(\mathfrak{K})$ -valued 1-form, this definition is independent of the choice of  $\nabla$  (cf. [JW13]).

If  $\mathfrak{K}$  has semisimple typical fibre  $\mathfrak{k}$ , then the isotypical ideals  $\mathfrak{k}_i^{m_i}$  in the decomposition  $\mathfrak{k} = \bigoplus_{i=1}^r \mathfrak{k}_i^{m_i}$  are  $\text{Aut}(\mathfrak{k})$ -invariant, so that we obtain a direct sum decomposition

$$\mathbb{V} = \bigoplus_{i=1}^r \mathbb{V}_i$$

of flat bundles.

If the ideal  $\mathfrak{k}_i$  is absolutely simple, which is always the case if  $\mathfrak{k}$  is a compact Lie algebra, then the structure group of  $\mathbb{V}_i$  reduces to  $S_{m_i}$ . In particular, if  $\mathfrak{k}$  is compact simple, then  $\mathbb{V}$  is simply the trivial line bundle  $M \times \mathbb{R} \rightarrow M$ .

If the ideal  $\mathfrak{k}_i$  possesses a complex structure, then the structure group of  $\mathbb{V}_i$  reduces to  $(\mathbb{Z}/2\mathbb{Z})^{m_i} \rtimes S_{m_i}$ . In particular, for  $\mathfrak{k}$  complex simple, the bundle  $\mathbb{V} \rightarrow M$  is the vector bundle with fibre  $\mathbb{C}$ , and  $\alpha \in \text{Aut}(\mathfrak{k})$  flips the complex structure on  $\mathbb{C}$  if and only if it flips the complex structure on  $\mathfrak{k}$ . If  $\mathfrak{K} = \text{Ad}(P)$  for a principal fibre bundle  $P \rightarrow M$  with complex simple structure group  $K$ , then  $\mathbb{V}$  is the trivial bundle  $M \times \mathbb{C} \rightarrow M$ .

## 5 Central extensions of gauge algebras

Let  $\mathfrak{g}$  be the compactly supported gauge algebra  $\Gamma_c(\mathfrak{K})$  for a Lie algebra bundle  $\mathfrak{K} \rightarrow M$  with semisimple fibres. In this section, we will classify all possible central extensions of  $\mathfrak{g} \rtimes_D \mathbb{R}$ , in other words, we will calculate the continuous second

Lie algebra cohomology  $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$  with trivial coefficients. In [JN16] we will examine which of these cocycles comes from a positive energy representation.

**Remark 5.1.** For a cocycle  $\omega$  on  $\mathfrak{g} \rtimes_D \mathbb{R}$ , the relation

$$\omega(D, [\xi, \eta]) = \omega(D\xi, \eta) + \omega(\xi, D\eta) \quad (11)$$

shows that  $i_D\omega$  measures the non-invariance of the restriction of  $\omega$  to  $\mathfrak{g} \times \mathfrak{g}$  under the derivation  $D$ . It also shows that, if the Lie algebra  $\mathfrak{g}$  is perfect, then the linear functional  $i_D\omega: \mathfrak{g} \rightarrow \mathbb{R}$  is completely determined by (11).

### 5.1 Definition of the 2-cocycles

We define 2-cocycles  $\omega_{\lambda, \nabla}$  on  $\mathfrak{g} \rtimes_D \mathbb{R}$  such that their classes span the cohomology group  $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ . They depend on a  $\mathbb{V}$ -valued 1-current  $\lambda \in \Omega_c^1(M, \mathbb{V})'$ , and on a Lie connection  $\nabla$  on  $\mathfrak{K}$ . Recall from Section 4 that  $\kappa: \mathfrak{k} \times \mathfrak{k} \rightarrow V(\mathfrak{k})$  is the universal invariant bilinear form of  $\mathfrak{k}$ , and  $\mathbb{V} \rightarrow M$  is the flat bundle with fibres  $\mathbb{V}_x = V(\mathfrak{K}_x)$ . In the important special case that  $\mathfrak{k}$  is compact simple,  $V(\mathfrak{k}) = \mathbb{R}$ ,  $\kappa$  is the Killing form, and  $\mathbb{V}$  is the trivial real line bundle.

A 1-current  $\lambda \in \Omega_c^1(M, \mathbb{V})'$  is said to be

(L1) *closed* if  $\lambda(dC_c^\infty(M, \mathbb{V})) = 0$ , and

(L2)  $\pi_*\mathbf{v}$ -*invariant* if  $\lambda(L_{\pi_*\mathbf{v}}\Omega_c^1(M, \mathbb{V})) = \{0\}$ .

Given a closed  $\pi_*\mathbf{v}$ -invariant current  $\lambda \in \Omega_c^1(M, \mathbb{V})'$ , we define the 2-cocycle  $\omega_{\lambda, \nabla}$  on  $\mathfrak{g} \rtimes_D \mathbb{R}$  by skew-symmetry and the equations

$$\omega_{\lambda, \nabla}(\xi, \eta) = \lambda(\kappa(\xi, \nabla\eta)), \quad (12)$$

$$\omega_{\lambda, \nabla}(D, \xi) = \lambda(\kappa(L_{\mathbf{v}}\nabla, \xi)), \quad (13)$$

where we write  $\xi$  for  $(\xi, 0) \in \mathfrak{g} \rtimes_D \mathbb{R}$  and  $D$  for  $(0, 1) \in \mathfrak{g} \rtimes_D \mathbb{R}$  as in (1). We define the  $\text{der}(\mathfrak{K})$ -valued 1-form  $L_{\mathbf{v}}\nabla \in \Omega^1(M, \text{der}(\mathfrak{K}))$  by

$$(L_{\mathbf{v}}\nabla)_w(\xi) = L_{\mathbf{v}}(\nabla\xi)_w - \nabla_w L_{\mathbf{v}}\xi = L_{\mathbf{v}}(\nabla_w\xi) - \nabla_w L_{\mathbf{v}}\xi - \nabla_{[\pi_*\mathbf{v}, w]}\xi \quad (14)$$

for all  $w \in \mathfrak{X}(M)$ ,  $\xi \in \Gamma(\mathfrak{K})$ . Since the fibres of  $\mathfrak{K} \rightarrow M$  are semisimple, all derivations are inner, so we can identify  $L_{\mathbf{v}}\nabla$  with an element of  $\Omega^1(M, \mathfrak{K})$ . Using the formulæ

$$d\kappa(\xi, \eta) = \kappa(\nabla\xi, \eta) + \kappa(\xi, \nabla\eta), \quad (15)$$

$$L_{\pi_*\mathbf{v}}\kappa(\xi, \eta) = \kappa(L_{\mathbf{v}}\xi, \eta) + \kappa(\xi, L_{\mathbf{v}}\eta), \quad (16)$$

$$L_{\mathbf{v}}(\nabla\xi) - \nabla L_{\mathbf{v}}\xi = [L_{\mathbf{v}}\nabla, \xi], \quad (17)$$

it is not difficult to check that  $\omega_{\lambda, \nabla}$  is a cocycle. Skew-symmetry follows from (15) and (L1). The vanishing of  $\delta\omega_{\lambda, \nabla}$  on  $\mathfrak{g}$  follows from (15), the derivation property of  $\nabla$  and invariance of  $\kappa$ . Finally,  $i_D\delta\omega_{\lambda, \nabla} = 0$  follows from skew-symmetry, (17), (16), (L2) and the invariance of  $\kappa$ .

Note that the class  $[\omega_{\lambda, \nabla}]$  in  $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$  depends only on  $\lambda$ , not on  $\nabla$ . Indeed, two connection 1-forms  $\nabla$  and  $\nabla'$  differ by  $A \in \Omega^1(M, \text{der}(\mathfrak{K}))$ . Using  $\text{der}(\mathfrak{K}) \simeq \mathfrak{K}$ , we find

$$\omega_{\lambda, \nabla'} - \omega_{\lambda, \nabla} = \delta \chi_A \quad \text{with} \quad \chi_A(\xi \oplus t) := \lambda(\kappa(A, \xi)).$$

## 5.2 Classification of central extensions

We now show that every continuous Lie algebra 2-cocycle on  $\mathfrak{g} \rtimes_D \mathbb{R}$  is cohomologous to one of the type  $\omega_{\lambda, \nabla}$  as defined in (12) and (13). The proof relies on a description of  $H^2(\mathfrak{g}, \mathbb{R})$  provided by the following theorem ([JW13, Prop. 1.1]).

**Theorem 5.2.** (Central extensions of gauge algebras) *Let  $\mathfrak{g}$  be the compactly supported gauge algebra  $\mathfrak{g} = \Gamma_c(\mathfrak{K})$  of a Lie algebra bundle  $\mathfrak{K} \rightarrow M$  with semisimple fibres. Then every continuous 2-cocycle is cohomologous to one of the form*

$$\psi_{\lambda, \nabla}(\xi, \eta) = \lambda(\kappa(\xi, \nabla \eta)),$$

where  $\lambda: \Omega_c^1(M, \mathbb{V}) \rightarrow \mathbb{R}$  is a continuous linear functional that vanishes on  $\text{d}\Omega_c^0(M, \mathbb{V})$ , and  $\nabla$  is a Lie connection on  $\mathfrak{K}$ . Two such cocycles  $\psi_{\lambda, \nabla}$  and  $\psi_{\lambda', \nabla'}$  are equivalent if and only if  $\lambda = \lambda'$ .

Using this, we classify the continuous central extensions of  $\mathfrak{g} \rtimes_D \mathbb{R}$ .

**Theorem 5.3.** (Central extensions of extended gauge algebras) *Let  $\mathcal{K} \rightarrow M$  be a bundle of Lie groups with semisimple fibres, equipped with a 1-parameter group of automorphisms with generator  $\mathbf{v} \in \mathfrak{X}(\mathcal{K})$ . Let  $\mathfrak{g} = \Gamma_c(\mathfrak{K})$  be the compactly supported gauge algebra, and let  $\mathfrak{g} \rtimes_D \mathbb{R}$  be the Lie algebra (7). Then the map  $\lambda \mapsto [\omega_{\lambda, \nabla}]$  induces an isomorphism*

$$\left( \Omega_c^1(M, \mathbb{V}) / (\text{d}\Omega_c^0(M, \mathbb{V}) + L_{\pi_* \mathbf{v}} \Omega_c^1(M, \mathbb{V})) \right)' \xrightarrow{\sim} H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$$

between the space of closed  $\pi_* \mathbf{v}$ -invariant  $\mathbb{V}$ -valued currents and  $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ .

*Proof.* Let  $\omega$  be a continuous 2-cocycle on  $\mathfrak{g} \rtimes_D \mathbb{R}$ . If  $i: \mathfrak{g} \hookrightarrow \mathfrak{g} \rtimes_D \mathbb{R}$  is the inclusion, then  $i^* \omega$  is a 2-cocycle on  $\mathfrak{g}$ . By Theorem 5.2 there exists a Lie connection  $\nabla$  and a continuous linear functional  $\varphi \in \mathfrak{g}'$  such that

$$i^* \omega(\xi, \eta) = \lambda(\kappa(\xi, \nabla \eta)) + \varphi([\xi, \eta]), \quad \text{where} \quad \lambda \in \Omega_c^1(M, \mathbb{V})'.$$

Using the cocycle property (cf. Rk. 5.1), we find

$$\omega(D, [\xi, \eta]) = i^* \omega(L_{\mathbf{v}} \xi, \eta) + i^* \omega(\xi, L_{\mathbf{v}} \eta) \tag{18}$$

and hence, using (16) and (17),

$$\begin{aligned} \omega(D, [\xi, \eta]) &= \lambda(\kappa(L_{\mathbf{v}} \xi, \nabla \eta) + \kappa(\xi, \nabla L_{\mathbf{v}} \eta)) + \varphi(L_{\mathbf{v}} [\xi, \eta]) \\ &= \lambda(L_{\pi_* \mathbf{v}} \kappa(\xi, \nabla \eta)) + \lambda(\kappa(L_{\mathbf{v}} \nabla, [\xi, \eta])) + \varphi(L_{\mathbf{v}} [\xi, \eta]). \end{aligned}$$

In particular,  $[\xi, \eta] = 0$  implies  $\lambda(L_{\pi_* \mathbf{v}} \kappa(\xi, \nabla \eta)) = 0$ .

Now fix a trivialisation  $\mathcal{K}|_U \simeq U \times K$  over an open subset  $U \subseteq M$ . It induces the corresponding trivialisation  $\mathbb{V}|_U \simeq U \times V(\mathfrak{k})$  of flat bundles. For  $f, g \in C_c^\infty(U)$  and  $X \in \mathfrak{k}$ , we consider  $\xi = fX$  and  $\eta = gX$  as commuting elements of  $\Gamma_c(\mathfrak{K})$ . With the local connection 1-form  $A \in \Omega^1(U, \mathfrak{k})$ , we then have

$$\kappa(\xi, \nabla \eta) = \kappa(fX, dg \cdot X + g[A, X]) = f dg \cdot \kappa(X, X).$$

Since  $[\xi, \eta] = 0$ , we find  $\lambda((L_{\pi_* \mathbf{v}} \beta \kappa(X, X))) = 0$  for all 1-forms  $\beta = f dg$  with  $f, g \in C_c^\infty(U)$ . As this holds for all  $X \in \mathfrak{k}$  and as  $\kappa(\mathfrak{k}, \mathfrak{k}) = V(\mathfrak{k})$ , we find  $\lambda(L_{\pi_* \mathbf{v}} \Omega_c^1(U, \mathbb{V})) = \{0\}$  by polarisation. Since  $\Omega_c^1(M, \mathbb{V}) = \sum_{i \in I} \Omega_c^1(U_i, \mathbb{V})$  for any trivialising open cover  $(U_i)_{i \in I}$  of  $M$ , we find  $\lambda(L_{\pi_* \mathbf{v}} \Omega_c^1(M, \mathbb{V})) = \{0\}$ .

Having established that  $\lambda$  is  $\pi_* \mathbf{v}$ -invariant, we may construct  $\omega_{\lambda, \nabla}$  according to (12) and (13). It then follows from the above that the difference

$$\Delta \omega := \omega - \omega_{\lambda, \alpha} + \delta \varphi^0,$$

where  $\varphi^0$  is an extension of  $\varphi$  to  $\mathfrak{g} \rtimes_D \mathbb{R}$ , satisfies  $i^* \Delta \omega = 0$ . Applying (18) to  $\Delta \omega$ , we see that  $\Delta \omega(D, [\mathfrak{g}, \mathfrak{g}]) = 0$  and hence that  $\Delta \omega = 0$  because  $\mathfrak{g}$  is perfect by [JW13, Prop. 2.4].

This shows surjectivity of the map  $\lambda \mapsto [\omega_{\lambda, \nabla}]$ . Injectivity follows because  $\omega_{\lambda, \nabla} = \delta \chi$  implies  $\omega_{\lambda, \nabla}|_{\mathfrak{g} \times \mathfrak{g}} = \delta(\chi|_{\mathfrak{g}})$ , hence  $\lambda = 0$  by Theorem 5.2.  $\square$

**Remark 5.4.** If the Lie connection  $\nabla$  on  $\mathfrak{K}$  can be chosen so as to make  $\mathbf{v} \in \mathfrak{X}(\mathcal{K})$  horizontal,  $\nabla_{\pi_* \mathbf{v}} \xi = L_{\mathbf{v}} \xi$  for all  $\xi \in \Gamma(\mathfrak{K})$ , then equation (14) shows that  $L_{\mathbf{v}} \nabla = i_{\pi_* \mathbf{v}} R$ , where  $R$  is the curvature of  $\nabla$ . For such connections, (13) is equivalent to

$$\omega_{\lambda, \nabla}(D, \xi) = \lambda(\kappa(i_{\pi_* \mathbf{v}} R, \xi)). \quad (19)$$

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